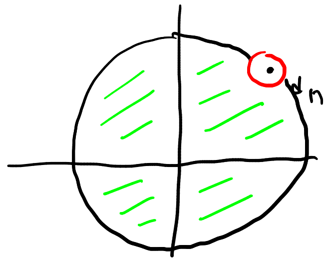
Topics in complex analysis. Lecture 3 (Sept 30, 2023)

Proof of Thm 2.2 $\qquad 0 < |d_1| \le |d_2| \le \dots$



Since q_n holomorphic on $B_{|d_n|}(0)$ we can write it locally as a power series. For all $z \in B_{|d_n|}(0)$

$$q_n(z) = \sum_{j=0}^{\infty} a_{nj} z^j \quad (\text{abs conv})$$

In particular this conv is unif (in z) on $B_{|d_n|/2}(0)$

Hence

$$\sup_{z \in B_{|d_n|/2}(0)} \Big| q_n(z) - \underbrace{\sum_{j=0}^{j_n} a_{nj} z^j}_{p_n(z)} \Big| \le 2^{-n}$$

Fix cpt set $K \subseteq \mathbb{C} \setminus S$. We claim $\sum_{n=1}^{\infty} \sup_{z \in K} |q_n(z) - p_n(z)| < \infty$

Since S is discrete, $|d_n| \to \infty$ as $n \to \infty$. Since K bdd $\exists n(K) \in \mathbb{N}$ $\forall n \ge n(K)$ $K \subseteq B_{|d_n|/2}(0)$. Then

$$\sum_{n \ge n(K)} \sup_{z \in K} |q_n(z) - p_n(z)| \le \sum_{n \ge n(K)} \sup_{z \in B_{|d_n|/2}(0)} |q_n(z) - p_n(z)| \le \frac{1}{2}$$

Since $q_1 - p_1, \dots, q_{n(K)-1} - p_{n(K)-1}$ are finitely many bounded functions, we get

$$\sum_{n=1}^{\infty} \sup_{z \in K} |q_n(z) - p_n(z)| < \infty$$

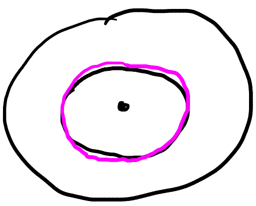
By Lemma 1.11 $f(z) = \sum_{n=1}^{\infty} [q_n(z) - p_n(z)]$ is holo

For the principal parts, given d_n choose $\rho > 0$ st $B_{2\rho}(d_n) \cap S = \emptyset$.

By Laurent series exp for the holo fct f the jth Laurent coeff at d_n is given by

$$a_j^n = \frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{f(z)}{(z-d_n)^{j+1}} \, dz$$

dom. cvg then



$$= \frac{1}{2\pi i} \sum_{k=1}^{\infty} \int_{\partial B_\rho(d_n)} \frac{q_k(z) - p_k(z)}{(z-d_n)^{j+1}} \, dz$$

For $j \leq -1$ the only integrand not holo on $B_{2\rho}(d_n)$ is $q_n(z)(z-d_n)^{-(j+1)}$. The other ones integrals vanish by Cauchy integral thm.

Hence

$$a_j = \underbrace{\frac{1}{2\pi i} \int_{\partial B_\rho(d_n)} \frac{q_n(z)}{(z-d_n)^{j+1}} \, dz}_{j\text{-th Laurent coeff of } q_n.}$$

□

Proof of Prop 2.6 Arguing similarly as above (pf of Thm 2.2), by Lemma 2.5 the truncated Laurent series' converge unif to q_n on the smaller annulus

$$A^2_{c_n} = \{ z \in \mathbb{C} : |z - c_n| \geq 2 |d_n - c_n| \}$$

Hence $\forall n \in \mathbb{N}$ $\exists k_n \in \mathbb{N}$ $\boxed{\sup_{z \in A^2_{c_n}} | q_n(z) - q_n^{k_n}(z) | \leq 2^{-n}}$

Fix a compact set $K \subseteq \mathbb{C} \setminus \bar{S}$ $\Rightarrow$ $\mathrm{dist}(\bar{S}, K) \geq \varepsilon$ for some $\varepsilon > 0$. Since $|d_n - c_n| \longrightarrow 0$ $\exists n(K) \in \mathbb{N}$ s.t. $\forall n \geq n(K)$ we have

$$K \subseteq A^2_{c_n} = \{ z \in \mathbb{C} : \overbrace{|z - c_n|}^{\geq \varepsilon} \geq 2 \underbrace{|d_n - c_n|}_{\longrightarrow 0} \}$$

In particular

$$\sum_{n \geq n(K)} \sup_{z \in K} | q_n(z) - q_n^{k_n}(z) |$$

$$\leq \sum_{n \geq n(K)} \sup_{z \in A^2_{c_n}} | q_n(z) - q_n^{k_n}(z) | < \infty$$

Since $q_1 - q_1^{k_1}, \dots, q_{n(K)-1} - q_{n(K)-1}^{k_{n(K)-1}}$ are bounded on K, we get $\displaystyle\sum_{n=1}^{\infty} \sup_{z \in K} |q_n(z) - q_n^{k_n}(z)| < \infty$

Hence by Lemma 1.11

$$f = \sum_{n=1}^{\infty} [q_n(z) - q_n^{k_n}(z)] \text{ is holo on } \mathbb{C} \setminus \bar{S}$$

Arguing as in the previous proof we show at every d_n the principal part of f is given by q_n. $\square$

Proof of Thm 2.9 wlog S' is nonempty (otherwise apply Thm 2.2) Let (cf Lemma 2.7)

$$S_1 = \{ z \in \mathbb{C} : |z| \operatorname{dist}(z, S') \geq 1 \} = \{ d_{n1} \}$$

closed (Lemma 2.7)

$$S_2 = \{ z \in \mathbb{C} : |z| \operatorname{dist}(z, S') < 1 \} = \{ d_{n2} \}$$

Since S_1 is closed we know $S_1' = \bar{S_1} \setminus S_1 = \emptyset$ and hence $S' = S_2'$.

1 Since S_1 is discrete, Thm 2.2 $\exists$ polynomials $p_{n1} : \mathbb{C} \to \mathbb{C}$ st

$$f_1 = \underbrace{\sum_{d_{n1} \in S_1} [q_{n1} - p_{n1}]}_{\text{locally normally cgt.}}$$

holo on $\mathbb{C} \setminus S_1$, and at each d_{n1} the princ part of f_1 is given by q_{n1}

2 Note by Lemma 2.8 the assumptions of Prop 2.6 are satisfied.

This yields $\exists$ holomorphic functions $h_{u_2} : \mathbb{C} \longrightarrow \mathbb{C}$

st $\underbrace{f_2 = \sum_{d_{u_2} \in S_2} [q_{u_2} - h_{u_2}]}_{\text{locally normal cvg.}}$ is holomorphic on $U \setminus S_2$ and at every d_{u_2} the principal part is given by q_{u_2}.

Taking $f = f_1 + f_2$ yields the desired concl $\square$